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1990 J. Phys. A: Math. Gen. 23 1973

(http://iopscience.iop.org/0305-4470/23/11/021)

J. Phys. A: Math. Gen. 23 (1990) 1973-1984. Printed in the UK

New spiky solitary waves, explosive modes and periodic progressive waves in a magnetised plasma

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Received 4 April 1989, in final form 26 January 1990

Abstract. A new class of nonlinear evolution equations presented here

$$\mathscr{A}(\mathrm{d}\phi/\mathrm{d}\eta)^2 = c_1\phi^2 + (\tfrac{4}{5})c_2\phi^{5/2} + (\tfrac{2}{3})c_3\phi^3 + (\tfrac{4}{7})c_4\phi^{7/2}$$

describes the one-dimensional wave propagation of ion acoustic waves in a magnetised plasma with trapped electrons. $\mathscr{A}$, c_1, c_2, c_3, c_4 are arbitrary constants. The remarkable feature of this equation is that a higher-order nonlinear term of $\phi^{7/2}$ appears in this equation. It is found that this equation has a new type of spiky solitary-wave solution, an explosive (bursting) solution and periodic progressive-wave solutions. The explosive solution is associated with the wave with the negative potential. Periodic progressive-wave solutions are reduced to the spiky solitary wave and the explosive solution under a certain condition. This theory may be applicable to explaining the behaviour of higher-order nonlinear waves in physical systems.

1. Introduction

Several studies of the nonlinear evolution equation have been made in the context of nonlinear plasma waves (Gardner and Morikawa 1965, Schamel 1973, Zakharov and Kuznetsov 1974, Torven 1986). In the situation where the nonlinear ion acoustic wave propagates in interplanetary space, spikey solitary waves and explosive (bursting) events are detected by satellites (Gurnett *et al* 1979, Temerin *et al* 1982, Kintner 1983). Although they are caused by the higher-order nonlinearity, little attention has been given to such progressive waves. Recently, nonlinear ion acoustic waves in unmagnetised plasmas have been studied as the subject of space plasma phenomena (Nejoh 1987a, 1988). However, higher-order nonlinear ion acoustic waves propagating obliquely to the magnetic field in magnetised plasmas have not yet been investigated. Zakharov (1988) stresses the importance of the higher-order nonlinearity with regard to the collapse of three-dimensional Langmuir waves, and mentions that the waves with negative energy have the explosive instability. The three-dimensional collapse includes one-dimensional phenomena, and has generality for physical phenomena. However, ion acoustic waves and the effect of trapped electrons are not considered.

In this paper, we investigate a nonlinear evolution equation

$$\mathscr{A}(\mathrm{d}\phi/\mathrm{d}\eta)^2 = c_1\phi^2 + \tfrac{4}{5}c_2\phi^{5/2} + \tfrac{2}{3}c_3\phi^3 + \tfrac{4}{7}c_4\phi^{7/2}$$

as a model for higher-order nonlinear wave propagation in a one-dimensional magnetised plasma with trapped electrons. $\mathscr{A}$, c_1, c_2, c_3 and c_4 are arbitrary parameters. This equation is characterised by the new nonlinear term $\phi^{7/2}$; this term has not yet been investigated. Nonlinear terms of this equation govern the behaviour of plasma waves where the nonlinear resonant (trapped) electron effect with the deviation from the isothermality is stronger than the effect of isothermal electrons. The former effect is expressed by the coefficients c_2 and c_4, and the latter the coefficient c_3 in this equation. Since such plasma waves usually exist in interplanetary space and are frequently observed (Lyons and Williams 1984) as energetic events, the study of this equation is significant in real situations. The nonlinear plasma response is described by fluid equations and by Poisson's equation under the quasineutral approximation. The quasineutral approximation allows us to study nonlinear waves with finite angles of propagation relative to the uniform magnetic field. The approximation requires the solutions to have slowly varying electric fields. It is known that the waves can exist only when their speed v_p lies in the range $v_i \cos\theta < v_p < v_i$, where v_i is the ion acoustic velocity $(\kappa T_e/M)^{1/2}$ and T_e is the electron temperature. We find that small-amplitude solitary waves occur only for $v_p = v_i \cos\theta$. It relates these waves to the slow ion acoustic wave mode in the case of Boltzman electrons (Lee and Kan 1981).

The principal object of this paper is to demonstrate a new type of spiky solitary wave, an explosive (bursting) solution and periodic progressive wave solutions of a class of one-dimensional higher-order nonlinear evolution equations. The author found a spiky solitary wave as the stationary solution of the derivative Boussinesq equation (Nejoh 1987b), but the present solitary wave is a new one. It will be expected that these solutions extend the scheme of well established nonlinear evolution equations and that this investigation may be applied to explaining the behaviour of larger-amplitude nonlinear waves propagating in plasmas.

The layout of this paper is as follows. In section 2 we consider the nonlinear propagation of one-dimensional ion acoustic waves propagating obliquely to the uniform magnetic field in the plasma with trapped electrons. The derivation of a model equation is presented from the basic equations. In section 3 the stationary solutions of this equation are shown. These are the spiky solitary wave and the explosive mode. In section 4, we derive periodic progressive wave solutions. These are described by the elliptic functions, and are reduced to the spiky solitary wave and the explosive solution in the limit of a certain parameter going to zero. The last section is devoted to concluding discussions.

2. A model equation

We begin by considering one-dimensional nonlinear ion acoustic waves propagating obliquely to the direction of the magnetic field in a uniformly magnetised plasma composed of free electrons, trapped electrons and cold ions. The scale length of the wave is assumed to be large compared to the ion cyclotron (gyro) radius and its propagation velocity v_p satisfies $v_{T_i} \ll v_p \ll v_{T_e}$, where v_{T_i} and v_{T_e} are the thermal velocities of ions and electrons, respectively. We assume that the small-but-finite amplitude ion acoustic wave propagates in the x direction and makes an angle θ to the direction of the magnetic field B, and that the magnetic field is in the x, z plane. Since the electron inertia is neglected for low-frequency oscillations of ion acoustic waves, the electron velocity is cancelled with the help of the equations of the electric field; the motions of electrons can therefore be ignored.

The fluid description used in this investigation can be justified by the fact that we are interested in the macroscopic, average nonlinear behaviour of the magnetised plasma rather than the microscopic properties, i.e. the motion of individual particles.

This justification can be further supported by the space and time scales involved in this nonlinear problem. We consider the low-frequency motion of the ion acoustic wave and assume that the longitudinal ion acoustic wave propagates parallel to the electric field. We hence observe the electrostatic ion acoustic wave in a magnetised plasma.

The dynamics of the ion fluid in this system is described by the equations of ion continuity, and of momentum transfer, and Poisson's equation. In non-dimensional form, they appear as

$$\frac{\partial n}{\partial t}+\nabla n\boldsymbol{v}=0 \tag{1a}$$

$$\left(\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\nabla\right)\boldsymbol{v}+\nabla\phi=-\frac{\nabla P}{n}+\boldsymbol{v}\times\boldsymbol{e}_{\mathrm{B}} \tag{1b}$$

$$\frac{\lambda_{\mathrm{D}}^2}{\rho^2}\nabla^2\phi+n-n_{\mathrm{e}}=0. \tag{1c}$$

Here, the ion density n, the ion pressure P, the time t and the velocity $\boldsymbol{v}$ are normalised by the background plasma density n_0, n_0T_{e}, the ion cyclotron frequency $\Omega(=eB/cM)$ and the ion acoustic velocity v_{i}, respectively. M and c are the ion mass and light velocity, respectively. The Debye length $\lambda_{\mathrm{D}}=(\varepsilon_0\kappa T_{\mathrm{e}}/n_0e^2)^{1/2}$ and the ion cyclotron radius $\rho=v_{\mathrm{i}}/\Omega$. T_{e} is the effective electron temperature given by $T_{\mathrm{e}}^{-1}=(1/e)(\partial n_{\mathrm{e}}/\partial\Phi)$ (for $\Phi=0$), where $\phi=\mathrm{e}\Phi/\kappa T_{\mathrm{e}}$ is the dimensionless potential. $\boldsymbol{e}_B$ is a unit vector along the magnetic field B.

In order to specify the components of equations (1) we use a double adiabatic equation of state for ions (Schmidt 1979) with parallel and perpendicular pressure components $p_{\|}=\alpha n^{\gamma_{\|}}$ and $p_{\perp}=\alpha n^{\gamma_{\perp}}$, where $\alpha=T_{\mathrm{i}}/T_{\mathrm{e}}$. $\gamma_{\|}$ and $\gamma_{\perp}$ are the parallel and perpendicular adiabatic indices which are taken to be 3 and 1, respectively. These values are determined by the correspondence between the linear fluid and kinetic dispersion relations in the quasineutral limit. $\lambda_{\mathrm{D}}/\rho\to0$. Since the electrons are assumed to be of zero inertia and strongly magnetised, their density can be found by the potential from the stationary Vlasov equation in the drift approximation (Akhiezer *et al* 1975).

Each components of equations (1), which satisfies the conditions mentioned above, can be written as

$$\frac{\partial n}{\partial t}+\frac{\partial}{\partial x}(nv_x)=0 \tag{2a}$$

$$\left(\frac{\partial}{\partial t}+v_x\frac{\partial}{\partial x}\right)v_x+\frac{\partial\phi}{\partial x}=v_y\sin\theta \tag{2b}$$

$$\left(\frac{\partial}{\partial t}+v_x\frac{\partial}{\partial x}\right)v_y=-v_x\sin\theta+v_z\cos\theta \tag{2c}$$

$$\left(\frac{\partial}{\partial x}+v_x\frac{\partial}{\partial x}\right)v_z=-v_y\cos\theta. \tag{2d}$$

The space coordinate x is normalised by the ion cyclotron radius ρ.

We use

$$\phi=\phi(\eta)$$

and a variable

$$\eta=x-st \tag{3}$$

where s denotes the velocity of the moving frame (Goswami and Bujarubarua 1986). Equations ($2a$) and ($2b$) yield

$$v_x = s\left(1-\frac{1}{n}\right) \qquad v_y = \frac{1}{\sin\theta}\left(1-s^2\frac{1}{n^3}\frac{\partial n}{\partial\phi}\right)\frac{\partial\phi}{\partial\eta}$$

respectively. Equations ($2c$) and ($2d$) reduce to

$$\frac{\partial v_y}{\partial\eta} = (n-1)\sin\theta - \frac{1}{s}nv_z\cos\theta \qquad \frac{\partial v_z}{\partial\eta} = \frac{1}{s}nv_y\cos\theta.$$

Then v_z is obtained as

$$v_z = \frac{\cot\theta}{s}\left(-s^2\left(1-\frac{1}{n}\right)+\int_0^\phi n(\phi)\,\mathrm{d}\phi\right),$$

by integrating

$$\int_0^\eta F\,\mathrm{d}\eta = \int_0^\phi F(\phi)\,\mathrm{d}\phi = \int_0^F F\frac{\mathrm{d}\phi}{\mathrm{d}F}\,\mathrm{d}F.$$

Eliminating the velocity components from equations (2) and using

$$\frac{\partial}{\partial\eta} = \frac{\partial\phi}{\partial\eta}\frac{\partial}{\partial\phi},$$

we obtain

$$\frac{1}{2}\frac{\partial}{\partial\phi}\left(\mathcal{A}(n)\frac{\partial\phi}{\partial\eta}\right)^2 = -\frac{\partial\mathcal{U}(\phi)}{\partial\phi} = \mathcal{A}(n)\left(n-1-\frac{n}{s^2}\cos^2\theta\int_0^\phi n(\phi)\,\mathrm{d}\phi\right) \tag{4}$$

where

$$\mathcal{A}(n) = 1-s^2\frac{1}{n^3}\frac{\partial n}{\partial\phi}.$$

$\mathcal{U}(\phi)$ denotes the potential function. We used the boundary conditions

$$n\to 1 \quad v_x\to 0 \qquad \text{at } \eta\to\infty.$$

We consider the plasma which satisfies the following conditions: the amplitude of the ion acoustic wave is small, the plasma is quasineutral and the ion cyclotron radius is much larger than the electron Debye length. In this case we can substitute the electron density for the ion density in equation (4). In order to study the effect of trapped electrons, we introduce the electron density derived from the Vlasov equation, which consists of free and trapped electrons (Schamel 1973). The electron density is expressed as

$$\begin{aligned} n_e(\phi) &= \int_{-\infty}^{\infty} f_e(x,v)\,\mathrm{d}v \\ &= \exp(\phi)\,\mathrm{erfc}(\phi^{1/2}) \\ &\quad + \frac{1}{|\beta|^{1/2}}\begin{cases} \exp(\beta\phi)\,\mathrm{erf}[(\beta\phi)^{1/2}] & \beta\geqslant 0 \\ \dfrac{2}{\pi^{1/2}}\exp\left(-[(-\beta\phi)^{1/2}]^2\int_0^{(-\beta\phi)^{1/2}}\exp(X^2)\,\mathrm{d}X\right) & \beta<0 \end{cases} \end{aligned} \tag{5}$$

where $f_e(x, v)$ and β indicate the electron distribution function and the ratio between the free and trapped electron temperature, T_f/T_e, respectively. We use the Maxwellian distribution for free and trapped electrons, and perform the Taylor expansion of equation (5) under the condition $\phi \ll 1$. Equation (5) thus reduces to

$$n_e(\phi) = 1 + \phi - \frac{4(1-\beta)}{3\pi^{1/2}}\phi^{3/2} + \frac{\phi^2}{2} - \frac{8(1-\beta^2)}{15\pi^{1/2}}\phi^{5/2} + \ldots \tag{6}$$

Considering up to the fifth term of equation (6), we rewrite the right-hand side of equation (4) as

$$-\frac{\partial \mathcal{U}(\phi)}{\partial \phi} = \mathcal{A}(c_1\phi + c_2\phi^{3/2} + c_3\phi^2 + c_4\phi^{5/2}), \tag{7}$$

where

$$c_1 = 1 - \frac{\cos^2\theta}{s^2} \qquad c_2 = -\frac{4(1-\beta)}{3\pi^{1/2}}$$

$$c_3 = \frac{1}{2}\left(1 - 3\frac{\cos^2\theta}{s^2}\right) \qquad c_4 = -\frac{4(1-\beta)}{15\pi^{1/2}}\left(2(1+\beta) - 7\frac{\cos^2\theta}{s^2}\right)$$

and

$$\mathcal{A} \simeq 1 - s^2.$$

Substituting equation (7) into equation (4) and integrating once, equation (4) becomes

$$(\phi_\eta)^2 = \frac{1}{\mathcal{A}}(c_1\phi^2 + \tfrac{4}{5}c_2\phi^{5/2} + \tfrac{2}{3}c_3\phi^3 + \tfrac{4}{7}c_4\phi^{7/2}), \tag{8}$$

under the boundary conditions

$$\phi, \phi_\eta, \phi_{\eta\eta} \to 0 \qquad \text{at } |\eta| \to \infty \tag{9}$$

where the subscript denotes partial differentiation.

3. Spiky solitary waves and explosive solutions

In order to solve equation (8), we set

$$\phi(\eta) = \psi^2(\eta). \tag{10}$$

Equation (8) is then transformed to

$$\begin{aligned}(\psi_\eta)^2 &= \psi^2(a_1 - a_2\psi - a_3\psi^2 - a_4\psi^3) \\ &= -\mathcal{V}(\psi) \\ &= 0\end{aligned} \tag{11}$$

where the parameters a_1, a_2, a_3 and a_4 stand for

$$a_1 = \frac{c_1}{4\mathcal{A}} \qquad a_2 = -\frac{c_2}{5\mathcal{A}} \qquad a_3 = -\frac{c_3}{6\mathcal{A}} \qquad a_4 = -\frac{c_4}{7\mathcal{A}}.$$

We observe that equation (11) is an equation which determines the trajectory of a particle with unit mass when the potential $\mathcal{V}(\psi)$ is given.

Now we are going to study the solitary-wave solutions and explosive solutions that follow from equation (11). We transform equation (11) to

$$(\psi_\eta)^2 = A\psi^2(\varphi_0 - \psi)^3 \tag{12}$$

provided that

$$A = a_4 \qquad 3a_2a_4 = a_3^2 \qquad s = -3^4\frac{a_1a_4^2}{a_3^3} \qquad \varphi_0 = -\frac{a_3}{3a_4}.$$

If $a_1>0$, $a_2>0$, $a_3<0$, $a_4>0$ and $A>0$, then $s>0$ and $\varphi_0>0$. We show the relation between ψ and $\mathscr{V}(\psi)$ in figure 1. The solutions which satisfy equations (11) and (12) exist in the upper region of the horizontal (ψ) axis in figure 1.

Integrating equation (12) and using the boundary conditions

$$\psi,\ \psi_\eta,\ \psi_{\eta\eta} \to 0 \qquad \text{at } |\eta| \to \infty \tag{13}$$

we obtain

$$\begin{aligned} \pm\tfrac{1}{2}A^{1/2}\varphi_0^{3/2}(\eta-\eta_0) &= -\left(\frac{\varphi_0}{\varphi_0-\psi(\eta)}\right)^{1/2} + \tanh^{-1}\left(\frac{\varphi_0-\psi(\eta)}{\varphi_0}\right)^{1/2} \\ &\quad + \left(\frac{\varphi_0}{\varphi_0-\psi_0}\right)^{1/2} - \tanh^{-1}\left(\frac{\varphi_0-\psi_0}{\varphi_0}\right)^{1/2} \end{aligned} \tag{14}$$

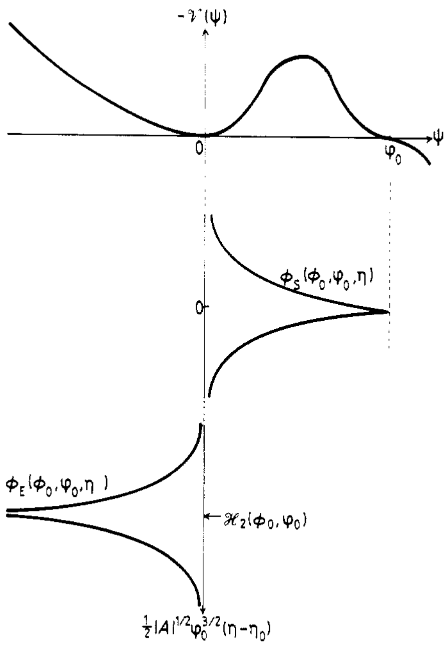

Figure 1. The potential function $\mathscr{V}(\psi)$ plotted against the coordinate ψ. $\phi_S(\phi_0, \varphi_0, \eta)$ and $\phi_E(\phi_0, \varphi_0, \eta)$ denote the spiky solitary wave corresponding to the region $0<\phi^{1/2}<\varphi_0$ and the explosive (bursting) solution corresponding to the region $\phi^{1/2}<0$, respectively. The arrow indicates the peak position $\mathscr{H}_2(\phi_0, \varphi_0)$ of $\phi_E(\phi_0, \varphi_0, \eta)$.

in the region $0<\psi(\eta)<\varphi_0$. Here ψ_0 denotes the maximum amplitude. Returning to equation (10), we obtain

$$\phi_S(\phi_0,\varphi_0,\eta)=\varphi_0^2\,\mathrm{sech}^4\left[\left(\frac{\varphi_0}{\varphi_0-\phi_S(\phi_0,\varphi_0,\eta)^{1/2}}\right)^{1/2}\pm\tfrac{1}{2}A^{1/2}\varphi_0^{3/2}(\eta-\eta_0)-\mathscr{H}_1(\phi_0,\varphi_0)\right.$$
$$\mathscr{H}_1(\phi_0,\varphi_0)=\left(\frac{\varphi_0}{\varphi_0-\phi_0^{1/2}}\right)^{1/2}-\tanh^{-1}\left(\frac{\varphi_0-\phi_0^{1/2}}{\varphi_0}\right)^{1/2} \qquad (15)$$

for $0<\phi(\phi_0,\varphi_0,\eta)^{1/2}<\varphi_0$, provided that $\phi_0=\psi_0^2$ from equation (10). We illustrate the wave profile of equation (15) in figure 1. Equation (15) forms a spiky solitary wave.

Similarly, we obtain

$$\begin{aligned}\pm\tfrac{1}{2}A^{1/2}\varphi_0^{3/2}(\eta-\eta_0)&\\ =-\left(\frac{\varphi_0}{\varphi_0-\psi(\eta)}\right)^{1/2}&+\coth^{-1}\left(\frac{\varphi_0-\psi(\eta)}{\varphi_0}\right)^{1/2}\\ +\left(\frac{\varphi_0}{\varphi_0-\psi_0}\right)^{1/2}&-\coth^{-1}\left(\frac{\varphi_0-\psi_0}{\varphi_0}\right)^{1/2}\end{aligned} \qquad (16)$$

in the region $\psi(\eta)<0$. Returning to the original expression, we have

$$\begin{aligned}\phi_E(\phi_0,\varphi_0,\eta)&\\ =\varphi_0^2\,\mathrm{cosech}^4&\left[\left(\frac{\varphi_0}{\varphi_0-\phi_E(\phi_0,\varphi_0,\eta)^{1/2}}\right)^{1/2}\pm\tfrac{1}{2}A^{1/2}\varphi_0^{3/2}(\eta-\eta_0)-\mathscr{H}_2(\phi_0,\varphi_0)\right]\end{aligned}$$
$$\mathscr{H}_2(\phi_0,\varphi_0)=\left(\frac{\varphi_0}{\varphi_0-\phi_0^{1/2}}\right)^{1/2}-\coth^{-1}\left(\frac{\varphi_0-\phi_0^{1/2}}{\varphi_0}\right)^{1/2} \qquad (17)$$

for $\phi(\phi_0,\varphi_0,\eta)^{1/2}<0$, where ϕ_0 is already determined. Equation (17) takes an explosive profile at $\pm\frac{1}{2}A^{1/2}\varphi_0^{3/2}(\eta-\eta_0)\to\mathscr{H}_2(\phi_0,\varphi_0)$, which is illustrated again in figure 1.

4. Periodic progressive wave solutions

In this section, returning to equation (11), we investigate stationary periodic progressive wave solutions.

We introduce the variable

$$v(\eta)=-\tfrac{1}{4}(a_4\psi+\tfrac{1}{3}a_3). \qquad (18)$$

By use of equation (18), equation (11) is transformed to

$$(v_\eta)^2=g_0^2(v-g_1)^2(4v^3-g_2v-g_3) \qquad (19)$$

where g_0, g_1, g_2 and g_3 are determined to be

$$g_0=\frac{4}{a_4}\qquad g_1=-\frac{a_3}{12}$$
$$g_2=\tfrac{1}{12}a_3^2-\tfrac{1}{4}a_2a_4\qquad g_3=8\left(\frac{a_3}{12}\right)^3-\tfrac{1}{48}a_2a_3a_4-a_1\left(\frac{a_4}{4}\right)^2.$$

Integrating equation (19), we obtain

$$\begin{aligned}\mathscr{K}(v)&=\int\frac{dv}{(v-g_1)(4v^3-g_2v-g_3)^{1/2}}\\ &=\pm g_0(\eta-\eta_0)\end{aligned} \qquad (20)$$

where η_0 is the integration constant. $\mathscr{K}(v)$ denotes the third kind of Weierstrass integral. Referring to the Weierstrass $\wp$ function with the parameters g_1, g_2 and g_3, we make the following transformations (Terasawa 1954):

$$\wp(f)=v(\eta) \qquad \mathrm{d}f=\frac{\mathrm{d}v}{(4v^3-g_2v-g_3)^{1/2}}$$
$$\wp(f_0)=g_1. \tag{21}$$

Here the $\wp(f)$ function is defined by

$$\wp(f)=\frac{1}{f^2}+\sum_{m,n}{}'\left(\frac{1}{(f-\Omega)^2}-\frac{1}{\Omega^2}\right) \tag{22}$$

where $\Sigma'_{m,n}$ denotes the sum over all m, n except the case where $m=n=0$ and Ω is the pole of $\wp(f)$. This pole is defined by

$$\Omega=2\omega_1 m+2\omega_3 n$$

where the integers $m, n=0, \pm1, \pm2, \ldots$. $2\omega_1$ and $2\omega_3$ are periods of the $\wp(f)$ function. ω_1 and ω_3 are defined by

$$\omega_1=\int_{e_1}^{\infty}\frac{\mathrm{d}v}{(4v^3-g_2v-g_3)^{1/2}} \qquad \omega_3=\mathrm{i}\int_{-\infty}^{e_3}\frac{\mathrm{d}v}{[-(4v^3-g_2v-g_3)]^{1/2}}$$

where e_1 and e_3 are determined from the equations

$$4v^3-g_2v-g_3=4(v-e_1)(v-e_2)(v-e_3),$$

and

$$e_1+e_2+e_3=0 \qquad e_2e_3+e_3e_1+e_1e_2=-\tfrac{1}{4}g_2$$
$$e_1^2+e_2^2+e_3^2=\tfrac{1}{2}g_2 \qquad e_1e_2e_3=\tfrac{1}{4}g_3.$$

Without loss of generality, we assume that

$$g_2^3-27g_3^2>0 \qquad e_1>e_2>e_3.$$

Using the $\zeta(f)$ function and $\sigma(f)$ function, we transform the integral $\mathscr{K}(v)$ to

$$\mathscr{K}(f)=\int\frac{\mathrm{d}f}{\wp(f)-\wp(f_0)}$$
$$=-\frac{1}{\partial\wp(f_0)/\partial f}\int[\zeta(f+f_0)-\zeta(f-f_0)-2\zeta(f_0)]\,\mathrm{d}f. \tag{23}$$

We note that $\zeta(f)$ function is connected with the $\sigma(f)$ function as follows:

$$\zeta(f)=\frac{\mathrm{d}}{\mathrm{d}f}\ln|\sigma(f)|. \tag{24}$$

Here $\sigma(f)$ is defined by

$$\sigma(f)=f\exp[\delta(f)]$$
$$=f\prod_{m,n}{}'\left[\left(1-\frac{f}{\Omega}\right)\exp\left(\frac{f}{\Omega}+\frac{f^2}{2\Omega^2}\right)\right] \tag{25}$$

provided that $\Pi'_{m,n}$ denotes the infinite product over all m, n except the case where $m=n=0$. We used the definition

$$\delta(f)=\sum_{m,n}'\left(\ln\left|\frac{f-\Omega}{-\Omega}\right|+\frac{f}{\Omega}+\frac{f^2}{2\Omega^2}\right).$$

On the other hand, the $\zeta(f)$ function is defined as

$$\begin{aligned}\zeta(f)&=\frac{1}{f}-\sum_{m,n}'\int_0^f\left(\frac{1}{(f-\Omega)^2}-\frac{1}{\Omega^2}\right)\mathrm{d}f\\&=\frac{1}{f}+\sum_{m,n}'\left(\frac{1}{f-\Omega}+\frac{1}{\Omega}+\frac{f}{\Omega^2}\right).\end{aligned}\tag{26}$$

Differentiating both sides of equation (26), we obtain

$$\wp(f)=-\frac{\mathrm{d}}{\mathrm{d}f}\zeta(f)\tag{27}$$

from equation (22). Substituting equation (23) into equation (27), we get

$$\wp(f)=-\frac{\mathrm{d}^2}{\mathrm{d}f^2}\ln|\sigma(f)|.\tag{28}$$

We therefore obtain

$$\mathscr{H}(f)=-\frac{1}{\partial\wp(f_0)/\partial f}\ln\left|\frac{1}{\sigma^2(f_0)}\frac{\sigma(f+f_0)}{\sigma(f-f_0)}\right|\tag{29}$$

from equations (23), (24) and (28).

Substituting equation (25) into equation (29) under the condition $f_0>0$, we have the equations

$$f_1(\eta)=-f_0\tanh\left(\tfrac{1}{2}(\pm g_0)\frac{\partial\wp(f_0)}{\partial f}(\eta-\eta_0+\eta_c)+\mathscr{Q}(f(\eta))\right)\tag{30}$$

in the region $-f_0<f<f_0$, and

$$f_2(\eta)=-f_0\tanh^{-1}\left(\tfrac{1}{2}(\pm g_0)\frac{\partial\wp(f_0)}{\partial f}(\eta-\eta_0+\eta_c)+\mathscr{Q}(f(\eta))\right)\tag{31}$$

in the region $-f_0>0$, or $f>f_0$, where

$$\mathscr{Q}(f(\eta))=\tfrac{1}{2}[\delta(f+f_0)-\delta(f-f_0)]\tag{32}$$

and

$$\eta_c=\frac{1}{(\pm g_0)\partial\wp(f_0)/\partial f}\ln[\sigma^2(f_0)].$$

From equations (18) and (21), $\psi(\eta)$ is reduced to

$$\psi(\eta)=-\frac{4}{a_4}\left(\wp(f(\eta))+\frac{a_3}{12}\right).\tag{33}$$

We determine the periodic-wave solutions as $\psi_1(\eta)$ and $\psi_2(\eta)$ corresponding to the functions $f_1(\eta)$ and $f_2(\eta)$, respectively, and thereby equation (33) is reduced to

$$\psi_1(\eta)=-\frac{4}{a_4}\left(\frac{a_3}{12}+\wp(f_1(\eta))\right)\tag{34}$$

and

$$\psi_2(\eta) = -\frac{4}{a_4}\left(\frac{a_3}{12} + \wp(f_2(\eta))\right). \tag{35}$$

In order to express the $\wp(f)$ function in terms of Jacobi's $\mathrm{sn}(f)$ function, we apply (Toda 1976)

$$v_1(\eta) = v_1(f_1(\eta)) = \wp(f_1(\eta)) = e_3 + (e_2 - e_3) f_1^2(\eta) \tag{36}$$

and

$$f_1(\eta) = \int_{e_3}^{v_1} \frac{\mathrm{d}v_1}{(4v_1^3 - g_2 v - g_3)^{1/2}}$$
$$= \frac{1}{(e_1 - e_3)^{1/2}} \int_0^{f_1} \frac{\mathrm{d}f_1}{[(1 - f_1^2)(1 - k^2 f_1^2)]^{1/2}}, \tag{37}$$

to equation (34), provided that

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3}. \tag{38}$$

From the relation

$$\int_0^{f_1} \frac{\mathrm{d}f_1}{[(1 - f_1^2)(1 - k^2 f_1^2)]^{1/2}} = \mathrm{sn}^{-1}(f_1(\eta, k)) \tag{39}$$

we have

$$f_1(\eta, k) = \mathrm{sn}[(e_1 - e_3)^{1/2} f_1(\eta), k]. \tag{40}$$

Substituting equation (40) into equation (36), we obtain

$$v_1(f_1(\eta, k)) = \wp(f_1(\eta), k) = e_3 + (e_2 - e_3)\,\mathrm{sn}^2[(e_1 - e_3)^{1/2} f_1(\eta), k]. \tag{41}$$

Similarly, making the transformation

$$v_2(\eta) = v_2(f_2(\eta)) = \wp(f_2(\eta)) = e_3 + \frac{e_1 - e_3}{f_2^2(\eta)}, \tag{42}$$

and

$$f_2(\eta) = -\int_{v_2}^{\infty} \frac{\mathrm{d}v_2}{(4v_2^3 - g_2 v_2 - g_3)^{1/2}}$$
$$= -\frac{1}{(e_1 - e_3)^{1/2}} \int_0^{f_2} \frac{\mathrm{d}f_2}{[(1 - f_2)^2 (1 - k^2 f_2^2)]^{1/2}} \tag{43}$$

we have

$$f_2(\eta, k) = -\mathrm{sn}[(e_1 - e_3)^{1/2} f_2(\eta), k]. \tag{44}$$

Hence $v_2[f_2(\eta, k)]$ is expressed as

$$v_2[f_2(\eta, k)] = \wp[f_2(\eta), k]$$
$$= e_3 + \frac{e_1 - e_3}{\mathrm{sn}^2[(e_1 - e_3)^{1/2} f_2(\eta), k]}. \tag{45}$$

Returning to equation (10) and using equations (41) and (45), we express $\phi(\eta, k)$ by Jacobi's $\mathrm{sn}(f(\eta), k)$ function. Determining the periodic-wave solutions as $\phi_1(\eta, k)$ and $\phi_2(\eta, k)$ corresponding to the function $\psi_1(\eta)$ and $\psi_2(\eta)$, respectively, we finally obtain

$$\phi_1(\eta, k) = \left(\frac{4}{a_4}\right)^2 \left(e_3 + \frac{a_3}{12} + (e_2 - e_3)\mathrm{sn}^2[(e_1 - e_3)^{1/2} f_1(\eta), k]\right)^2 \tag{46}$$

and

$$\phi_2(\eta, k) = \left(\frac{4}{a_4}\right)^2 \left(e_3 + \frac{a_3}{12} + \frac{e_1 - e_3}{\mathrm{sn}^2[(e_1 - e_3)^{1/2} f_2(\eta), k]}\right)^2. \tag{47}$$

Equations (46) and (47) imply the periodic progressive wave solutions corresponding to the region $0 < \phi^{1/2} < \varphi_0$ and $\phi^{1/2} < 0$ in figure 1, respectively.

If we take the limit $k \to 1$ in equation (46), we obtain the spiky solitary-wave solution (15). In addition, the same limit $k \to 1$ in equation (47) yields the explosive solution (17). It is because $\mathrm{sn}[f(\eta, k)] \to \tanh[f(\eta), 1]$ in the limit $k \to 1$.

5. Concluding discussion

We have presented a new nonlinear evolution equation derived from the one-dimensional nonlinear ion acoustic wave propagating obliquely to the magnetic field in a magnetised plasma with trapped electrons. This equation has a new higher-order nonlinear term. Equation (8) describes the behaviour of the plasma physical system in which the higher-order nonlinearity competes with the dispersion effect. The stationary solutions of this equation are obtained due to the higher-order nonlinearity, and bear the spiky solitary-wave solution, the explosive solution and periodic progressive-wave solution. It should be noted that the explosive solution eventually will make the ordering in the derivation of the higher-order nonlinear evolution equation break down. In addition, the explosive solution is associated with the wave with negative potential. The periodic progressive-wave solutions are reduced to the spiky solitary-wave solution and the explosive solution, in the limit $k \to 1$. It is worthwhile to note that the formation of these solutions remarkably depends on the signs of the parameters c_1, c_2, c_3 and c_4, and that the transformation from equation (11) to equation (12) holds only under special conditions on the parameters a_1, a_2, a_3 and a_4. These new solutions appear to make a step forward in the general scheme of nonlinear normal modes for long waves. It is worth noticing that, although the present solutions are derived for the one-dimensional potential, these results are similar to those derived in the three-dimensional collapse of Langmuir waves (Zakharov 1988) and to those of the electric field structure of collapsing wavepackets in Langmuir turbulence (Newman *et al* 1989).

The present results have crucial importance when we discuss nonlinear ion acoustic waves in plasmas where the nonlinear resonant (trapped) electron effect with the deviation from the isothermality is stronger than the effect of the isothermal electrons. Stationary-wave solutions presented here have to satisfy the conditions $c_1 > 0$, $c_2 > 0$, $c_3 < 0$, $c_4 > 0$ and $s > 0$; these solutions are valid only under these conditions. In this case, it is evident that $a_1 > 0$, $a_2 > 0$, $a_3 < 0$ and $a_4 > 0$. When the plasma parameters satisfy

$$0.166 T_t < T_f < T_t \qquad \cos^{-1} 3^{-1/2} < \theta < \pi/2$$

then $0 < s < 1$ and $\phi_0 > 0$ hold. The author therefore stresses that these conditions are highly relevant for the given physical problem. In actual situations, spiky ion acoustic solitary waves and solar radio burst events associated with trapped electrons are frequently observed in interplanetary space (Gurnett *et al* 1979, Alfvén 1981, Temerin *et al* 1982, Lin *et al* 1986). Hence, referring to the present spiky solitary waves, explosive (bursting) modes and periodic progressive waves, we can understand the properties of ion acoustic waves where the higher-order nonlinearity is essential in space plasmas.

Although the author has not examined the application of these results to a specific observational result, this investigation is important in discussing the higher-order nonlinear wave modes. This theory is therefore applicable to another nonlinear progressive waves in physical systems.

Acknowledgment

This work is partly supported by a Grant-in-Aid of the Hachinohe Institute of Technology.

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